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SHORT COMMUNICATIONS

STRESS ANALYSIS OF A SAND PARTICLE WITH INTERFACE IN CEMENT PASTE UNDER UNIAXIAL LOADING

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SUMMARY

In this paper, the stress states of a sand particle (or aggregate) with an interface layer in a cement paste (or mortar) subjected to uniaxial compression or tension are studied. This is a dual layer inclusion problem. The general analytical solutions of stresses and deformations are obtained in closed form, and the solutions of several special cases including the sand (or aggregate) treated as rigid body and as a hole as well as when the thickness of the interface layer approaches zero are also given.

KEY WORDS: elasticity; stress analysis; inclusion; interface; sand

1. INTRODUCTION

It is well-known that the concrete material consists of aggregates, sands and cement paste or mortar. Experimental studies and analytical analyses have shown that non-linear, softening and brittle fracture behaviours of concrete are all caused by the development of microcracks in the material under loading. It was found that these cracks are always formed and developed first at the interfaces between sand particles and cement paste or aggregates and mortar (Figure 1(a)).

To understand the basic strength, elastic-plastic property and softening behaviour of concrete materials, we must study the fundamental mechanisms of crack developments. To this end, it is necessary to determine the stress distribution at the interface between sand particle (or aggregate) and cement paste (or mortar) and to understand the influence of various geometrical and physical parameters of the particles and the interface on stress distribution, from which we may proceed to study the cause of crack formation and rule of crack development.

Herein, we shall treat the sand particle (or aggregate) as an idealized circular inclusion and take the interface between the sand particle and the cement paste as a concentric thin layer with different material property. Thus, this becomes a dual inclusion problem with a circular nucleus and a circular tube-shaped thin layer under uniaxial tension or compression (Figure 1(b)).

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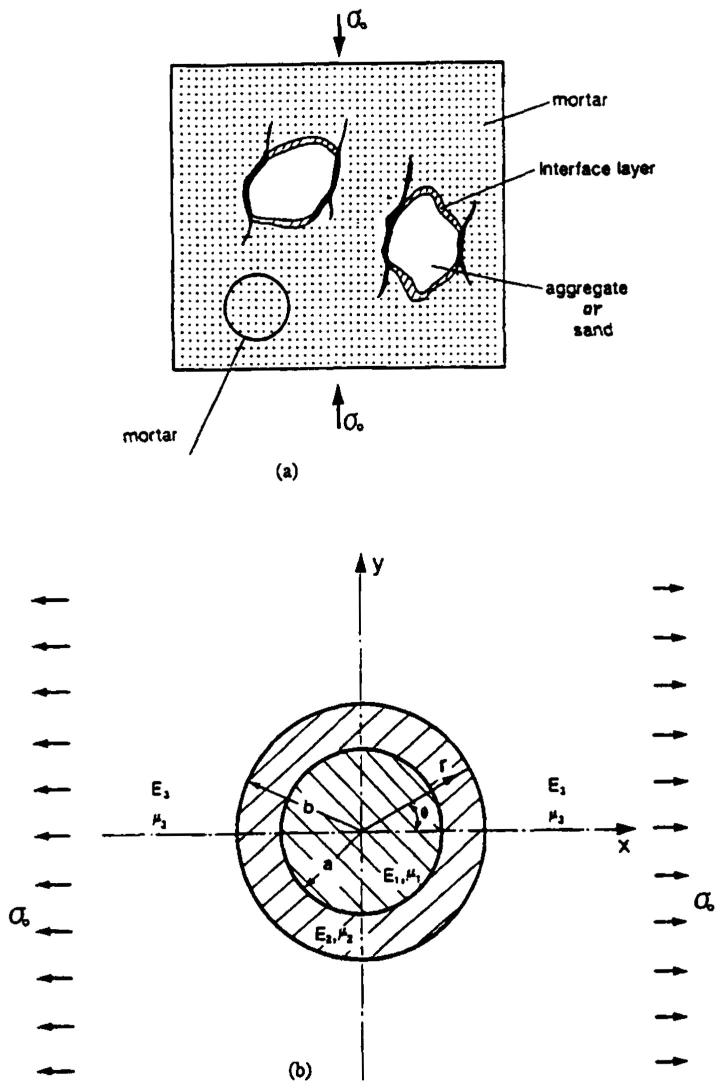

Figure 1. The model of microstructure of concrete

For the inclusion problems, a considerable effort has been made. Goodier[1] solved stress states of an infinite thin plate with a heterogeneous insertion subjected to uniaxial tension. The stress state of an infinite body having two inclusions was solved by Goree[2] and others. England,[3] Viola and Piva,[4] Ihara and Shaw[5] and Yamaguchi[6] solved one or several inclusion problems in finite region using different methods.

For inclusion problems with interface, Christensen and Lo[7] solved three phase sphere and cylinder models in pure shear case for obtaining the effective shear modulus of composites. Benveniste *et al.*[8] obtained the stress field of six fundamental loading cases for coated cylindrical fibres, and then extended to the case of coated orthotropic fibres.[9] Further extension was made by Pagano and Tandon to obtain the elastic response of multi-directional coated fibre composites.[10]

In the composite material research with coated fibres, attention has been focused on the effective elastic moduli, but paid little attention to the stress concentration and stress distribution around the inclusion. Furthermore, the constants to be determined in the stress field solutions are given in terms of simultaneous linear equations and are not expressed in explicit forms.

In concrete micromechanic research, the explicit closed-form formulas for stress concentration and stress distribution around the interface are of great interest. In this paper, we shall obtain these formulas. The present solution procedure is also applicable for the cases of multi-layer elastic inclusion problems. For several special cases for which the sand is treated either as a rigid, or an elastic body or a hole, the corresponding stress states are also obtained in closed form.

2. BASIC SOLUTIONS

Let a and b be, respectively, the inner and outer radii of the interface, and E_1, μ_1, E_2, μ_2 and E_3, μ_3, respectively, elastic constants of the inclusion, interface layer and the cement paste (Figures 1(b)). We shall adopt the polar co-ordinates r, θ and find its stress solution subjected to a uniaxial tension stress σ_0 at infinity.

Owing to symmetry to x and y axes (Figure 1(b)), only the $\cos n\theta$ ($n = 0, 2, 4, 6, \dots$) terms are included in the general solution of stress function,[11,12] we take stress function as

$$\Phi = A_0 \ln r + C_0 r^2 + [A_2 r^2 + B_2 r^4 + C_2 r^{-2} + D_2]\cos 2\theta + [A_4 r^4 + B_4 r^6 + C_4 r^{-4} + D_4 r^{-2}]\cos 4\theta + \cdots \tag{1}$$

and obtain

$$\begin{aligned} \sigma_r &= A_0 \frac{1}{r^2} + 2C_0 + [-2A_2 - 6C_2 r^{-4} - 4D_2 r^{-2}]\cos 2\theta + \cdots \\ \sigma_\theta &= -A_0(1/r^2) + 2C_0 + [2A_2 + 12B_2 r^2 + 6C_2 r^{-4}]\cos 2\theta + \cdots \\ \tau_{r\theta} &= [2A_2 + 6B_2 r^2 - 6C_2 r^{-4} - 2D_2 r^{-2}]\sin 2\theta + \cdots \end{aligned} \tag{2}$$

$$\begin{aligned} u &= \frac{1}{E}\Big\{ -A_0(1+\mu)\frac{1}{r} + C_0 2(1-\mu)r + [-2(1+\mu)A_2 r \\ &\quad - 4\mu B_2 r^3 + 2(1+\mu)C_2 r^{-3} + 4D_2 r^{-1}]\cos 2\theta + \cdots \Big\} \\ v &= \frac{1}{E}\{[2(1+\mu)A_2 r + 2(3+\mu)B_2 r^3 + 2(1+\mu)C_2 r^{-3} - 2(1-\mu)D_2 r^{-1}]\sin 2\theta + \cdots\} \end{aligned} \tag{3}$$

where A_0, C_0, A_2, B_2, $\dots$, C_4, D_4 are constants that must be determined from the boundary conditions of this problem.

3. DETERMINATION OF ARBITRARY CONSTANTS

3.1. *The boundary conditions*

As $r \to \infty$,

$$\begin{aligned} \sigma_r'' &= (\sigma_0/2)(1 + \cos 2\theta) \\ \sigma_\theta'' &= (\sigma_0/2)(1 - \cos 2\theta) \\ \tau_{r\theta}'' &= -(\sigma_0/2)\sin 2\theta \end{aligned} \tag{4}$$

and stress components are finite, we obtain

$$B''_2 = A''_4 = B''_4 = \cdots = 0$$

$$C''_0 = \frac{\sigma_0}{4}, \qquad A''_2 = -\frac{\sigma_0}{4} \tag{5}$$

As $r = 0$, $u' = v' = 0$ and stress components are finite. We have $A'_0 = C'_2 = D'_2 = C'_4 = D'_4 = \cdots = 0$.

3.2. The continuity conditions at $r = b$ and $r = a$

From continuity conditions at $r = b$ and $r = a$, we have

$\sigma_r = \sigma''_r$

$$A_0 \frac{1}{b^2} + 2C_0 + \left[-2A_2 - C_2 \frac{6}{b^4} - D_2 \frac{4}{b^2} \right] \cos 2\theta + \cdots$$
$$= A''_0 \frac{1}{b^2} + \frac{\sigma_0}{2} + \left[\frac{\sigma_0}{2} - C''_2 \frac{6}{b^4} - D''_2 \frac{4}{b^2} \right] \cos 2\theta + \cdots$$

$\tau_{r\theta} = \tau''_{r\theta}$

$$\left[2A_2 + 6B_2 b^2 - C_2 \frac{6}{b^4} - D_2 \frac{2}{b^2} \right] \sin 2\theta + \cdots$$
$$= \left[-\frac{\sigma_0}{2} - C''_2 \frac{6}{b^4} - D''_2 \frac{2}{b^2} \right] \sin 2\theta + \cdots$$

$u = u''$

$$\frac{1}{E_2} \left\{ -A_0(1 + \mu_2)\frac{1}{b} + C_0 2(1 - \mu_2)b \right.$$
$$\left. + \left[-A_2 2(1 + \mu_2)b - B_2 4\mu_2 b^3 + C_2 2(1 + \mu_2)\frac{1}{b^3} + D_2 \frac{4}{b} \right] \cos 2\theta + \cdots \right\}$$
$$= \frac{1}{E_3} \left\{ -A''_0(1 + \mu_3)\frac{1}{b} + \frac{\sigma_0}{2}(1 - \mu_3)b \right.$$
$$\left. + \left[\frac{\sigma_0}{2}(1 + \mu_3)b + C''_2 2(1 + \mu_3)\frac{1}{b^3} + D''_2 \frac{4}{b} \right] \cos 2\theta + \cdots \right\}$$

$v = v''$

$$\frac{1}{E} \left\{ \left[A_2 2(1 + \mu_2)b + B_2 2(3 + \mu_2)b^3 + C_2 2(1 + \mu_2)\frac{1}{b^3} - D_2 2(1 - \mu_2)\frac{1}{b} \right] \sin 2\theta + \cdots \right\}$$
$$= \frac{1}{E_3} \left\{ \left[-\frac{\sigma_0}{2}(1 + \mu_3)b + C''_2 2(1 + \mu_3)\frac{1}{b^3} - D''_2 2(1 - \mu_3)\frac{1}{b} \right] \sin 2\theta + \cdots \right\}$$

$\sigma'_r = \sigma_r$

$$2C'_0 - 2A'_2 \cos 2\theta + \cdots = A_0 \frac{1}{a^2} + 2C_0 + \left[-2A_2 - 6C_2 \frac{1}{a^4} - D_2 \frac{4}{a^2} \right] \cos 2\theta + \cdots$$

$\tau'_{r\theta} = \tau_{r\theta}$:

$$[2A'_2 + 6B'_2a^2]\sin 2\theta + \cdots = \left[2A_2 + 6B_2a^2 - C_2\frac{6}{a^4} - D_2\frac{2}{a^2}\right]\sin 2\theta + \cdots$$

$u' = u$

$$\frac{1}{E_1}\{C'_0 2(1-\mu_1)a + [-A'_2 2(1+\mu_1)a - B'_2 4\mu_1 a^3]\cos 2\theta + \cdots\}$$

$$= \frac{1}{E_2}\left\{-A_0(1+\mu_2)\frac{1}{a} + C_0 2(1-\mu_2)a\right.$$

$$\left. + \left[-A_2 2(1+\mu_2)a - B_2 4\mu_2 a^3 + C_2 2(1+\mu_2)\frac{1}{a^3} + D_2\frac{4}{a}\right]\cos 2\theta + \cdots\right\}$$

$v' = v$

$$\frac{1}{E_1}\{[A'_2 2(1+\mu_1)a + B'_2 2(3+\mu_1)a^3]\sin 2\theta + \cdots\}$$

$$= \frac{1}{E_2}\left\{\left[A_2 2(1+\mu_2)a + B_2 2(3+\mu_2)a^3 + C_2 2(1+\mu_2)\frac{1}{a^3} - D_2 2(1-\mu_2)\frac{1}{a}\right]\sin 2\theta + \cdots\right\} \tag{6}$$

In the above quantities with prime, with double prime and without prime are the ones for the inclusion ($r \leqslant a$), for the cement paste ($r \geqslant b$) and for interface layer ($a \leqslant r \leqslant b$), respectively.

4. SOLUTIONS

In order to satisfy equation (6), we must take the coefficients of the terms with $\cos n\theta$ and $\sin n\theta$ ($n = 2, 4, 6, \ldots$, respectively) to zero, and then solve these equations to determine constants. After determining all constants, substitute them into equation (2) and (3), we obtain finally the stresses and displacements in the inclusion, cement paste and interface layer as

(1) *In the inclusion* ($r \leqslant a$)

$$\sigma'_r = \left(1 + \frac{1}{m}\right)\frac{1}{a^2}A_0 - 2A'_2\cos 2\theta$$

$$\sigma'_\theta = \left(1 + \frac{1}{m}\right)\frac{1}{a^2}A_0 + (2A'_2 + 12B'_2r^2)\cos 2\theta$$

$$\tau'_{r\theta} = (2A'_2 + 6B'_2r^2)\sin 2\theta \tag{7}$$

$$u' = \frac{1}{E_1}\left\{(1-\mu_1)\left(1 + \frac{1}{m}\right)A_0\frac{r}{a^2} - [2(1+\mu_1)A'_2r + 4\mu_1B'_2r^3]\cos 2\theta\right\}$$

$$v' = \frac{1}{E_1}\{2(1+\mu_1)A'_2r + 2(3+\mu_1)B'_2r^3\}\sin 2\theta$$

where

$$A_0 = \frac{\sigma_0}{\lambda_3\left[\frac{1-\mu_2}{a^2m} - \frac{1+\mu_2}{b^2}\right] + \frac{1+\mu_3}{b^2}\left[1 + \frac{b^2}{a^2m}\right]}$$

$$A'_2 = \frac{\sigma_0}{\Delta}\left\{-3a^2(b^2-a^2) - \frac{b^6}{a^2m_4}\left(1 + \frac{1}{m_3}\right) + m_1a^4\left(\frac{1}{m_3} - \frac{3b^2}{a^2} + 4\right)\right\} \tag{8}$$

$$B'_2 = \frac{2\sigma_0}{\Delta}(1+m_1)(b^2-a^2)$$

(2) *In cement paste* ($r \geqslant b$)

$$\sigma''_r = \left(1 + \frac{b^2}{a^2m}\right)A_0\frac{1}{r^2} + \frac{\sigma_0}{2}\left(1 - \frac{b^2}{r^2}\right) + \left[\frac{\sigma_0}{2} - 6C''_2r^{-4} - 4D''_2r^{-2}\right]\cos 2\theta$$

$$\sigma''_\theta = -\left(1 + \frac{b^2}{a^2m}\right)A_0\frac{1}{r^2} + \frac{\sigma_0}{2}\left(1 + \frac{b^2}{r^2}\right) + \left[-\frac{\sigma_0}{2} + 6C''_2r^{-4}\right]\cos 2\theta$$

$$\tau''_{r\theta} = \left[-\frac{\sigma_0}{2} - 6C''_2r^{-4} - 2D''_2r^{-2}\right]\sin 2\theta \tag{9}$$

$$u'' = \frac{1}{E_3}\left\{-(1+\mu_3)\left(1 + \frac{b^2}{a^2m}\right)A_0\frac{1}{r} + \frac{\sigma_0}{2}\left[(1-\mu_3)r + (1+\mu_3)\frac{b^2}{r}\right]\right.$$
$$\left. + \left[\frac{\sigma_0}{2}(1+\mu_3)r + C''_2 2(1+\mu_3)r^{-3} + D''_2 4r^{-1}\right]\cos 2\theta\right\}$$

$$v'' = \frac{1}{E_3}\left\{\left[-\frac{\sigma_0}{2}(1+\mu_3)r + C''_2 2(1+\mu_3)r^{-3} - D''_2 2(1-\mu_3)r^{-1}\right]\sin 2\theta\right\}$$

where

$$C''_2 = \frac{\sigma_0}{\Delta}\left\{(b^2-a^2)(3a^2b^4 - 4b^6) + \frac{1}{m_4}b^6\left[\frac{b^4}{a^2m_3} + a^2\right] - m_1a^6\left[\frac{b^4}{a^2m_3} + b^2\right]\right\} - \frac{1}{4}\sigma_0b^4 \tag{10}$$

$$D''_2 = \frac{\sigma_0}{\Delta}\left\{6b^2(b^2-a^2)^2 - 2\left(\frac{b^6}{m_4} - a^6m_1\right)\left[1 + \frac{b^2}{a^2m_3}\right]\right\} + \frac{1}{2}\sigma_0b^2$$

(3) *In the interface layer* ($a \leqslant r \leqslant b$)

$$\sigma_r = \left(\frac{1}{r^2} + \frac{1}{a^2m}\right)A_0 - [2A_2 + 6C_2r^{-4} + 4D_2r^{-2}]\cos 2\theta$$

$$\sigma_\theta = \left(-\frac{1}{r^2} + \frac{1}{a^2m}\right)A_0 + [2A_2 + 12B_2r^2 + 6C_2r^{-4}]\cos 2\theta$$

$$\tau_{r\theta} = [2A_2 + 6B_2r^2 - 6C_2r^{-4} - 2D_2r^{-2}]\sin 2\theta \tag{11}$$

$$u = \frac{1}{E_2}\left\{A_0\left[-(1+\mu_2)\frac{1}{r} + \frac{r}{a^2m}(1-\mu_2)\right] + \left[-2(1+\mu_2)A_2r\right.\right.$$
$$\left.\left. - 4\mu_2B_2r^3 + 2(1+\mu_2)C_2r^{-3} + 4D_2r^{-1}\right]\cos 2\theta\right\}$$

$$v = \frac{1}{E_2}\{2(1+\mu_2)A_2r + 2(3+\mu_2)B_2r^3 + 2(1+\mu_2)C_2r^{-3} - 2(1-\mu_2)D_2r^{-1}\}\sin 2\theta$$

where

$$A_2 = \frac{\sigma_0}{\Delta}\left\{ - 3(b^2 - a^2)a^2 - \frac{1}{a^2 m_3}\left(\frac{b^6}{m_4} - a^6 m_1\right)\right\}$$

$$B_2 = \frac{2\sigma_0}{\Delta}(b^2 - a^2)$$

$$C_2 = \frac{\sigma_0}{\Delta}\left\{a^2 b^6 \frac{1}{m_4} - a^6 b^2 m_1\right\} \tag{12}$$

$$D_2 = \frac{-2\sigma_0}{\Delta}\left\{b^6 \frac{1}{m_4} - a^6 m_1\right\}$$

$$\Delta = -[\lambda_3(1 + \mu_2) + 3 - \mu_3]\left\{\left[b^6 \frac{1}{m_4} - a^6 m_1\right]\left(\frac{m_2}{b^2} - \frac{1}{a^2 m_3}\right) + 3(b^2 - a^2)^2\right\}$$

in which

$$\lambda_1 = E_1/E_2, \qquad \lambda_3 = E_3/E_2, \qquad m = \frac{((1 - \mu_2)/E_2) - ((1 - \mu_1)/E_1)}{((1 + \mu_2)/(E_2) + ((1 - \mu_1)/E_1)}$$

$$m_1 = \frac{3 - \mu_2 - (3 - \mu_1)/\lambda_1}{1 + \mu_2 + (3 - \mu_1)/\lambda_1} \qquad m_2 = \frac{3 - \mu_2 - (3 - \mu_3)/\lambda_3}{1 + \mu_2 + (3 - \mu_3)/\lambda_3} \tag{13}$$

$$m_3 = \frac{1 + \mu_2 - (1 + \mu_1)/\lambda_1}{3 - \mu_2 + (1 + \mu_1)/\lambda_1} \qquad m_4 = \frac{1 + \mu_2 - (1 + \mu_3)/\lambda_3}{3 - \mu_2 + (1 + \mu_3)/\lambda_3}$$

This is the general solution of a sand particle (or aggregate) with an interface layer in a cement paste (or mortar) (dual layer inclusion problem) under uniaxial loading. This solution is in closed form. For some special cases (for examples $E_1 = E_2$; $E_2 = E_3$, etc.), there are some terms with infinitely large values in equations (7)–(12). For these cases, we may let $E_1 \approx E_2$; $E_2 \approx E_3$, etc.

5. NUMERICAL EXAMPLES

5.1. The inclusion and interface layer as a hole ($E_1 = E_2 = 0$, $\mu_1 = \mu_2 = 0$)

This is the case for a hole in the concrete. From equations (7)–(12), we obtain the stresses and displacements in the mortar ($r \geqslant b$) as

$$\sigma_r'' = \frac{\sigma_0}{2}\left(1 - \frac{b^2}{r^2}\right) + \left[\frac{\sigma_0}{2} + \frac{3}{2}\sigma_0\frac{b^4}{r^4} - 2\sigma_0\frac{b^2}{r^2}\right]\cos 2\theta$$

$$\sigma_\theta'' = \frac{\sigma_0}{2}\left(1 + \frac{b^2}{r^2}\right) + \left[-\frac{\sigma_0}{2} - \frac{3}{2}\sigma_0\frac{b^4}{r^4}\right]\cos 2\theta$$

$$\tau_{r\theta}'' = \left[-\frac{\sigma_0}{2} + \frac{3}{2}\sigma_0\frac{b^4}{r^4} - \sigma_0\frac{b^2}{r^2}\right]\sin 2\theta \tag{14}$$

$$u'' = \frac{1}{E_3}\left\{\frac{b^2}{2}\sigma_0(1 + \mu_3)\frac{1}{r} + \frac{\sigma_0}{2}(1 - \mu_3)r + \left[\frac{\sigma_0}{2}(1 + \mu_3)r - \frac{\sigma_0}{2}(1 + \mu_3)\frac{b^4}{r^3} + 2\sigma_0\frac{b^2}{r}\right]\cos 2\theta\right\}$$

$$v'' = \frac{1}{E_3}\left\{-\frac{\sigma_0}{2}(1 + \mu_3)r - \frac{\sigma_0}{2}(1 + \mu_3)\frac{b^4}{r^3} - \sigma_0(1 - \mu_3)\frac{b^2}{r}\right\}\sin 2\theta$$

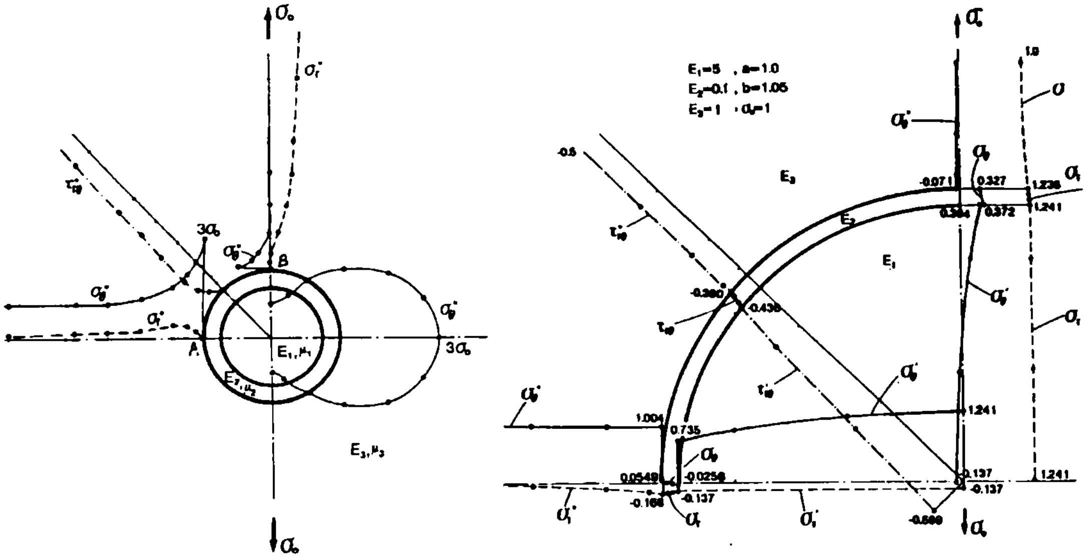

Figure 2. The stress distribution of microstructure of concrete. (a) A hole case; (b) the case of soft interface

This result corresponds exactly to the case of a infinitely large plate with one hole under uniaxial load.[1] The stress distribution in the mortar is shown in Figure 2(a).

5.2. Inclusion as a rigid body ($E_1 \to \infty$)

Assuming $E_2 = E_3$, $\mu_2 = \mu_3$, we obtain from equations (7)–(12):

In the inclusion

$$\begin{aligned} \sigma_r' &= \frac{\sigma_0}{2}(1+m) + \frac{\sigma_0}{2}(1+m_3)\cos 2\theta \\ \sigma_\theta' &= \frac{\sigma_0}{2}(1+m) - \frac{\sigma_0}{2}(1+m_3)\cos 2\theta \\ \tau_{r\theta}' &= -\frac{\sigma_0}{2}(1+m_3)\sin 2\theta \\ u' &= 0, \quad v' = 0 \end{aligned} \tag{15}$$

In the interface layer and mortar

$$\begin{aligned} \sigma_r'' &= \frac{\sigma_0}{2}\left(1 + m\frac{a^2}{r^2}\right) + \frac{\sigma_0}{2}\left[1 - 3m_3\frac{a^4}{r^4} + 4m_3\frac{a^2}{r^2}\right]\cos 2\theta \\ \sigma_\theta'' &= \frac{\sigma_0}{2}\left(1 - m\frac{a^2}{r^2}\right) + \frac{\sigma_0}{2}\left[-1 + 3m_3\frac{a^4}{r^4}\right]\cos 2\theta \\ \tau_{r\theta}'' &= \frac{\sigma_0}{2}\left[-1 - 3m_3\frac{a^4}{r^4} + 2m_3\frac{a^2}{r^2}\right]\sin 2\theta \end{aligned} \tag{16}$$

$$u'' = \frac{\sigma_0}{2E_0}\left\{ -(1+\mu_2)m\frac{a^2}{r} + (1-\mu_2)r + \left[(1+\mu_2)r + m_3(1+\mu_2)\frac{a^4}{r^3} - 4m_3\frac{a^2}{r}\right]\cos 2\theta \right\}$$

$$v'' = \frac{\sigma_0}{2E_0}\left\{ -(1+\mu_2)r + m_3(1+\mu_2)\frac{a^4}{r^3} + 2m_3(1-\mu_2)\frac{a^2}{r} \right\}\sin 2\theta$$

These results correspond exactly to those of a infinite plate with one rigid inclusion under uniaxial load.[1] The stress distribution in the sand–interface–cement paste is shown in Figure 3. In the case of rigid inclusion, stresses are independent of E, and they are only function of μ.

5.3. No interface layer ($a = b$)

Assuming $a = b$, $E_2 = E_3$ and $\mu_2 = \mu_3$, from equations (7)–(12), we have

in inclusion

$$\sigma'_r = \frac{\sigma_0}{2}(1+m) + \frac{\sigma_0}{2}(1+m_3)\cos 2\theta$$

$$\sigma'_\theta = \frac{\sigma_0}{2}(1+m) - \frac{\sigma_0}{2}(1+m_3)\cos 2\theta$$

$$\tau'_{r\theta} = -\frac{\sigma_0}{2}(1+m_3)\sin 2\theta \tag{17}$$

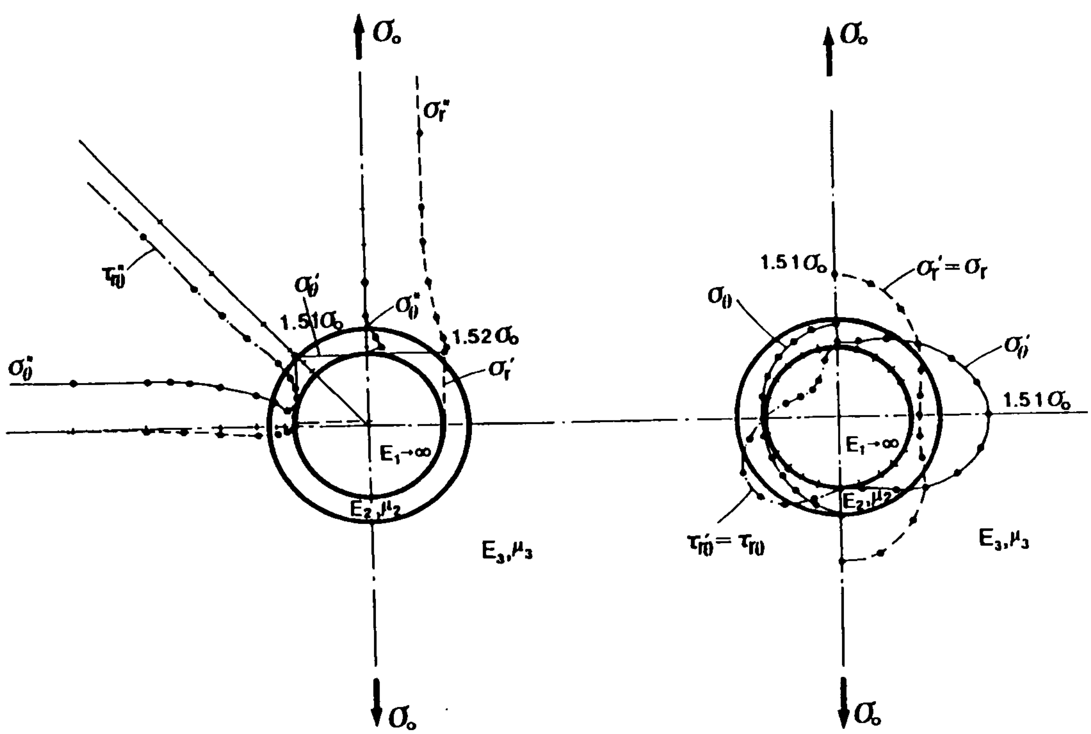

Figure 3. Rigid inclusion case ($E_1 \to \infty$, $E_2 = E_3$, $\mu_2 = \mu_3$)

$$u' = \frac{1}{E_1}\left\{\frac{\sigma_0}{2}(1-\mu_1)(1+m)r + \frac{\sigma_0}{2}(1+\mu_1)(1+m_3)r\cos 2\theta\right\}$$

$$v' = \frac{1}{E_1}\left\{-\frac{\sigma_0}{2}(1+\mu_1)(1+m_3)r\sin 2\theta\right\}$$

In mortar

$$\sigma_r'' = \frac{\sigma_0}{2}m\frac{b^2}{r^2} + \frac{\sigma_0}{2} + \left[\frac{\sigma_0}{2} - \frac{3\sigma_0}{2}m_3\frac{b^4}{r^4} + 2\sigma_0 m_3\frac{b^2}{r^2}\right]\cos 2\theta$$

$$\sigma_\theta' = -\frac{\sigma_0}{2}m\frac{b^2}{r^2} + \frac{\sigma_0}{2} + \left[-\frac{\sigma_0}{2} + \frac{3\sigma_0}{2}m_3\frac{b^4}{r^4}\right]\cos 2\theta$$

$$\tau_{r\theta}'' = \left[-\frac{\sigma_0}{2} - \frac{3\sigma_0}{2}m_3\frac{b^4}{r^4} + \sigma_0 m_3\frac{b^2}{r^2}\right]\sin 2\theta \tag{18}$$

$$u'' = \frac{1}{E_3}\left\{-\frac{\sigma_0}{2}(1+\mu_3)m\frac{b^2}{r} + \frac{\sigma_0}{2}(1-\mu_3)r\right.$$

$$\left.+\left[\frac{\sigma_0}{2}(1+\mu_3)r + \frac{\sigma_0}{2}m_3(1+\mu_3)\frac{b^4}{r^3} - 2\sigma_0 m_3\frac{b^2}{r}\right]\cos 2\theta\right\}$$

$$v'' = \frac{1}{E_3}\left\{-\frac{\sigma_0}{2}(1+\mu_3)r + \frac{\sigma_0}{2}(1+\mu_3)m_3\frac{b^4}{r^3} + \sigma_0(1-\mu_3)m_3\frac{b^2}{r}\right\}\sin 2\theta$$

In fact, this is a solution for an elastic inclusion problem as shown in Figure 4.

5.4 *Softer interface*

Assuming $E_1 = 5$, $E_2 = 0{\cdot}1$, $E_3 = 1$, $a = 1{\cdot}0$, $b = 1{\cdot}05$, $\sigma_0 = 1$, the stress distribution in the sand–interface–cement paste is shown in Figure 2(b). This is the case of soft interface.

6. CONCLUSION

In this paper, the stress function method was used to obtain the analytical solution for the stress distribution of a composite model consists of a sand particle with an interface layer in a cement paste (dual layer inclusion problem). This is a general solution in closed form with all constants explicitly determined. With this solution, we are now ready to make parametric study of various stress distributions and stress concentrations of the dual layer inclusion problem with geometrical and physical parameters appropriate for concrete materials. This will be described in a subsequent paper.[13]

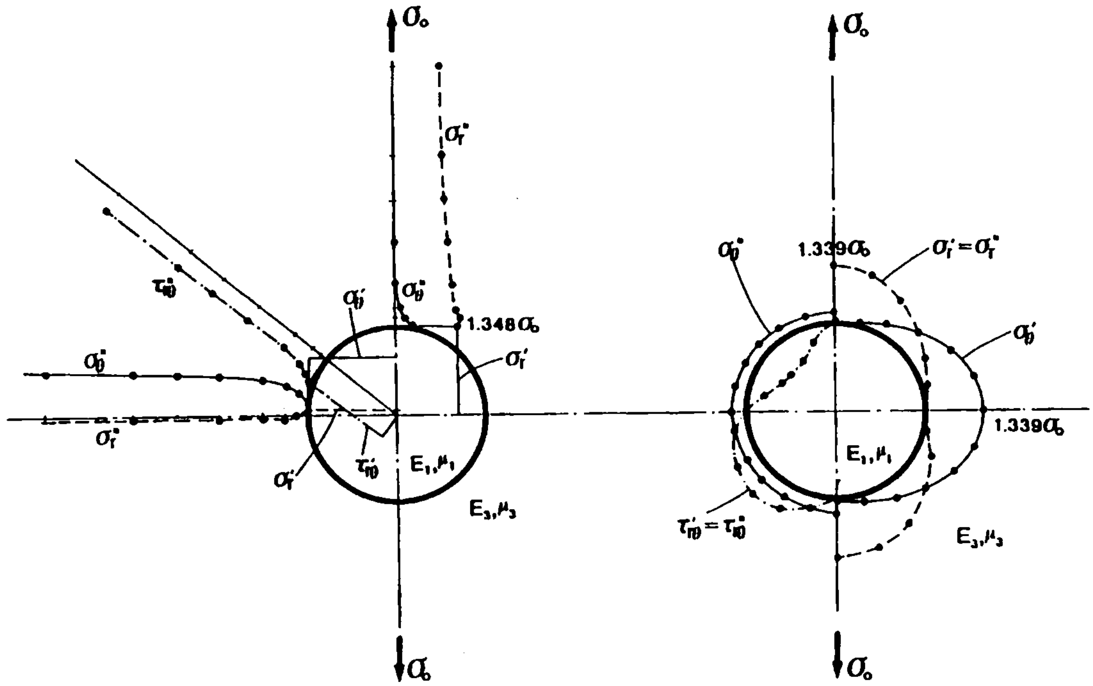

Figure 4. Elastic inclusion case ($a = b$, $E_1/E_3 = 4_1$, $\mu_1 = \mu_3$)

Several special numerical examples are also given here, from which we discover that the rigidity of sand particle and interface layer has a significant influence on its stress distribution of mortar material.

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